In the last lecture, we were deriving the forces that lead to vertical acceleration:

$$\frac{Dw}{Dt} = \frac{\rho'}{\rho_0^2}\frac{\partial P_0}{\partial z} - \frac{1}{\rho_0}\frac{\partial P'}{\partial z}$$

Recall that hydrostatic balance $\frac{\partial P_0}{\partial z} = -\rho_0 g \longleftrightarrow \frac{1}{\rho_0}\frac{\partial P_0}{\partial z} = -g$

Using hydrostatic balance, we get the following:

$$\frac{Dw}{Dt} = -\frac{\rho'}{\rho_0} g - \frac{1}{\rho_0}\frac{\partial P'}{\partial z}$$

Define buoyancy $B = -g\frac{\rho'}{\rho_0}$ $\rho' = \rho - \rho_0$

$$\frac{Dw}{Dt} = \underbrace{B}_{\text{Buoyancy}} \underbrace{- \frac{1}{\rho_0}\frac{\partial P'}{\partial z}}_{\text{non-hydrostatic PGF due to e.g. weather systems}}$$

B tells you that there's upward acceleration if $\rho' < 0$

If the density of a parcel is less than the environment (env. is the hydrostatic atm. with $\rho = \rho_0$)

* Let's assume that a rising parcel has the same pressure as the env. i.e. assume that $P = P_0$ $P' = 0$

$$B = -g\frac{\rho'}{\rho_0} = -g\frac{\rho - \rho_0}{\rho_0}$$

Use ideal gas law $P_0 = \rho R_d T$ (ignoring virtual effect)

Plug ideal gas law to B, and after some work obtain:

$$B \simeq g\frac{T - T_0}{T_0}$$

parcel that is warmer than env. rises. Colder parcel sinks.

We have:

$$\frac{Dw}{Dt} = g\frac{T - T_0}{T_0}$$

Let's assume that the env. has a constant lapse rate Γ, so that

$$T_0 = T_{0s} - \Gamma z$$

In a dry atmosphere, parcels rise at the dry adiabatic lapse rate

$$T = T_s - \Gamma_d z \qquad \Gamma_d = \frac{g}{c_p}$$

Back to mom. eqn.

$$\frac{Dw}{Dt} = g \frac{\cancel{T_s} - \Gamma_d z - (\cancel{T_{0s}} - \Gamma z)}{T_{0s} - \cancel{\Gamma z}}$$

assume that $T_{0s} \gg \Gamma z$
assume $T_{0s} = T_s$

$$= g \frac{(\Gamma - \Gamma_d) z}{T_{0s}}$$

Recall that $w = \frac{Dz}{Dt}$

Define $N^2 = -g \frac{(\Gamma - \Gamma_d)}{T_{0s}}$

$$\frac{D^2 z}{Dt^2} = N^2 z$$

↑ second-order ODE with the following solution:

$$z(t) = c_1 e^{iNt} + c_2 e^{-iNt}$$

N = Brunt-Vaisala freq.

Case (1): $\Gamma > \Gamma_d$ $\longrightarrow$ $N \Rightarrow$ real N^2 = positive

Solution becomes $z(t) = A\cos(Nt) + B\sin(Nt)$

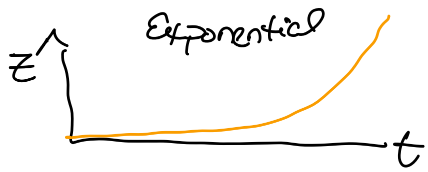
Case (2) $\Gamma < \Gamma_d$ $N \Rightarrow$ imaginary N^2 = negative

$N = iN_0$ making i explicit
$i = \sqrt{-1}$

$$z(t) = c_1 e^{i(iN_0)t} + c_2 e^{-i(iN_0)t}$$

$$= \underline{c_1} e^{-N_0 t} + \underline{c_2} e^{N_0 t}$$

init. conds.



Depending on initial conditions, soln. either grows or decays with time.

What happens when you have a saturated atmosphere?

$$T_0 = T_{0s} - \Gamma z$$
$$T = T_s - \Gamma_m z$$

* Now because RH = 100%, any lifting will result in condensation.

Mom. equation becomes:

$$\frac{D^2 z}{Dt^2} = g \frac{(\Gamma - \Gamma_m) z}{T_{0s}}$$

when $\Gamma > \Gamma_m$ the soln is unstable

$$N_m^2 = -g \frac{(\Gamma - \Gamma_m)}{T_{0s}}$$

we want N_m^2 to be negative to have instability

When $\Gamma' > \Gamma_m$ the environment coolds down faster than the moist adiabatic lapse rate