Buoyancy and stability

The vertical mom. eqn. is written as:

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad (1)$$

We know hydrostatic balances is the leading-order balance in the atmosphere. Acceleration is smaller

$$\frac{Dw}{Dt} << g$$

To elucidate the acceleration, we separate p and ρ into hydrostatic and perturbation components.

$$\underbrace{p = p_0}_{\text{hydrostatic}} + \underbrace{p'}_{\text{perturbation}} \qquad \rho = \rho_0 + \rho' \qquad \rho_0 >> \rho' \qquad p_0 >> p'$$

We state that $\frac{-1}{\rho_0}\frac{\partial p_0}{\partial z} \equiv g$ (2) exactly equal/ by definition

Expand ρ and p in Eq. (1)

$$\frac{Dw}{Dt} = \frac{-1}{\rho_0+\rho'}\frac{\partial}{\partial z}(p_0+p') - g$$

$$\frac{1}{\rho_0}\left(\frac{1}{1+\rho'/\rho_0}\right) \simeq \frac{1}{\rho_0}\left(1-\frac{\rho'}{\rho_0}\right) = \frac{1}{\rho_0} - \frac{\rho'}{\rho_0^2}$$

Expand into Taylor Series

$$\frac{Dw}{Dt} = -\left(\frac{1}{\rho_0} - \frac{\rho'}{\rho_0^2}\right)\left(\frac{\partial p_0}{\partial z} + \frac{\partial p'}{\partial z}\right) - g$$

$$= -\left[\cancel{\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}} - \frac{\rho'}{\rho_0^2}\frac{\partial p_0}{\partial z} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \cancel{\frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial z}}\right] - \cancel{g}$$

cancel out b.c. of Eq. (2)

Smaller than the rest

$$\frac{Dw}{Dt} = \frac{\rho'}{\rho_0^2}\frac{\partial p_0}{\partial z} - \frac{1}{\rho_0}\frac{\partial p'}{\partial z}$$

$$= \underbrace{-\frac{\rho'}{\rho_0} g}_{\text{Buoyancy}} - \frac{1}{\rho_0}\frac{\partial p'}{\partial z}$$