

Last time we derived \dot{Q}_e based on the notion that \dot{Q}_e is conserved
 i.e. $\frac{D\dot{Q}_e}{Dt} = 0$ or $d\dot{Q}_e = 0$

Let's consider the general scenario

Define diabatic heating $\dot{Q} = L_v(c-e) + L_s(d-s) + \dot{Q}_r + \dot{J}_h + L_f(f-m)$

$$\dot{Q} = \frac{\delta q}{\delta t}$$

$c =$ condensation $e =$ evaporation
 $d =$ deposition $s =$ sublimation
 $\dot{J}_h =$ flux conv. of enthalpy
 $\dot{Q}_r =$ radiative heating
 $f =$ freezing $m =$ melting

$$L_v = 2.5 \times 10^6 \text{ J kg}^{-1}$$

$$L_s = L_v + L_f \approx 2.83 \times 10^6 \text{ J kg}^{-1}$$

$$L_f \approx 3.34 \times 10^5 \text{ J kg}^{-1}$$

Going back to the first law and expanding:

$$C_p \frac{DT}{Dt} - \alpha \frac{DP}{Dt} = L_v(c-e) + L_s(d-s) + \dot{Q}_r + \dot{J}_h + L_f(f-m)$$

Recall $\frac{Dq_v}{Dt} = S_g$ $S_g = -(c-e + d-s) + \dot{J}_g L_v^{-1}$

Expand the right-hand side:

$$C_p \frac{DT}{Dt} - \alpha \frac{DP}{Dt} = L_v(c-e + d-s) + L_f(f-m + d-s) + \dot{Q}_r + \dot{J}_h$$

$$C_p \frac{DT}{Dt} - \alpha \frac{DP}{Dt} = -L_v \frac{Dq_v}{Dt} + L_f(f-m + d-s) + \dot{Q}_r + \dot{J}_h + \dot{J}_g$$

Define $\dot{Q}_e = L_f(f-m + d-s) + \dot{Q}_r + \dot{J}_h + \dot{J}_g$
 Equivalent diabatic source

Using \dot{Q}_e and going back to the first law yields:

$$C_p \frac{DT}{Dt} - \alpha \frac{DP}{Dt} + L_v \frac{Dq_v}{Dt} = \dot{Q}_e$$

Let's return to infinitesimal form:

$$C_p dT - \alpha dp + L_v dq_v = \delta q_e$$

Let's divide by T and use ideal gas law $p\alpha = \frac{RdT}{P}$
 $\alpha = \frac{RdT}{P}$

$$C_p d \ln T - R d \ln p + L_v \frac{dq_v}{T} = \frac{\delta Q_e}{T}$$

$$C_p d \ln T - R d \ln p + L_v d \left(\frac{q_v}{T} \right) = \frac{\delta Q_e}{T}$$

We can bring all the constants within the differential:

$$d \left(C_p \ln T - R \ln p + L_v \frac{q_v}{T} \right) = \frac{\delta Q_e}{T}$$

$$\oint d \left(C_p \ln T - R \ln p + L_v \frac{q_v}{T} \right) = \oint \frac{\delta Q_e}{T} = 0$$

↑ these are all state vars
↑ is also a state var.

We have defined a new state variable

$$S_m = \frac{\delta Q_e}{T} = C_p \ln T - R \ln p + L_v \frac{q_v}{T} + \text{const}$$

Moist entropy

$$= C_p \ln \Theta + L_v \frac{q_v}{T} + \text{const}$$

Very important for HW4

HW4, problem 1, show that $dS_m = C_p d \ln \Theta$

Under moist adiabatic processes $dS_m = 0$

Moist entropy is conserved moist adiabatically.

A special form of moist entropy that can be obtained for a hydrostatic atmosphere

Return to first law: $C_p \frac{dT}{dt} - \alpha \frac{dp}{dt} + L_v \frac{dq_v}{dt} = \dot{Q}_e$

If atmosphere is hydrostatic $\frac{\partial p}{\partial z} = -\rho g \rightarrow \alpha dp = -g dz = -d\Phi$
 $\Phi = \text{geopotential}$

We can use the chain rule $\frac{dp}{dt} = \frac{dp}{dz} \frac{dz}{dt} = -\rho g \frac{dz}{dt} = -\rho \frac{d\Phi}{dt}$

We replace in 1st law $C_p \frac{dT}{dt} + \alpha \rho \frac{d\Phi}{dt} + L_v \frac{dq_v}{dt} = \dot{Q}_e$

$$C_p \frac{dT}{dt} + \frac{d\Phi}{dt} + L_v \frac{dq_v}{dt} = \dot{Q}_e$$

$$\frac{D}{Dt} (c_p T + \Phi + L_v q_v) = \dot{Q}_e$$

Define $m = c_p T + \Phi + L_v q_v$ Moist static energy (MSE)

MSE is widely used in climate science and tropical meteorology.

MSE, S_m and E_z are useful as summary vars. of moist processes, even when they're not conserved.

Moist adiabatic lapse rate

Dry adiabatic lapse rate

$$\frac{ds_s}{dz} = 0 \rightarrow \frac{ds_s}{dz} = \frac{d}{dz} (c_p T + \Phi)$$

$$s_s = MSE \rightarrow \frac{dT}{dz} = -\frac{g}{c_p} = -\Gamma_d$$

$$\Gamma_d = \frac{g}{c_p}$$

Moist adiabatic lapse rate

$$\frac{dm}{dz} = \frac{d}{dz} (c_p T + \Phi + L_v q_v)$$

After some algebra

$$\frac{dT}{dz} = -\Gamma_m$$

$$\Gamma_m = \Gamma_d \left(\frac{1 + \frac{L_v q_s}{R_d T}}{1 + \frac{L_v^2 q_v}{c_p R_v T^2}} \right)$$

In general $\Gamma_m < \Gamma_d$ because latent heat release from condensation partially cancels adiabatic cooling.