

AOS 630: Introduction to Atmospheric  
and Oceanic Physics  
Lecture 10 Fall 2021  
*The Clausius-Clapeyron Equation*

Ángel F. Adames-Corraliza  
angel.adamescorraliza@wisc.edu





# Announcements

Third Skew-T a day is due today

HW2 is due on Thursday.

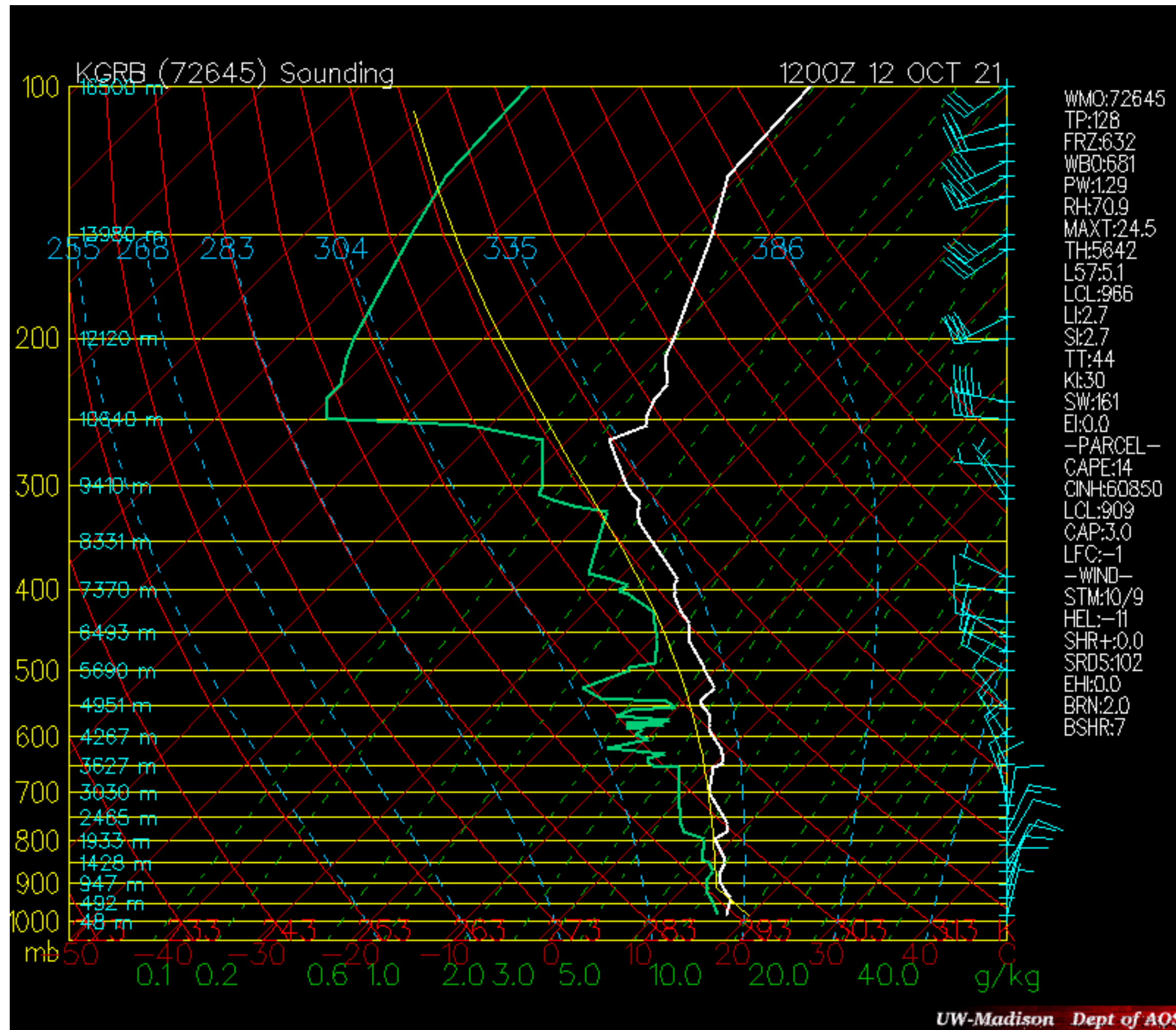
HW3 will be posted on Thursday as well.

# Skew-T a week #4

The green line in this plot is the dew point (discussed today).

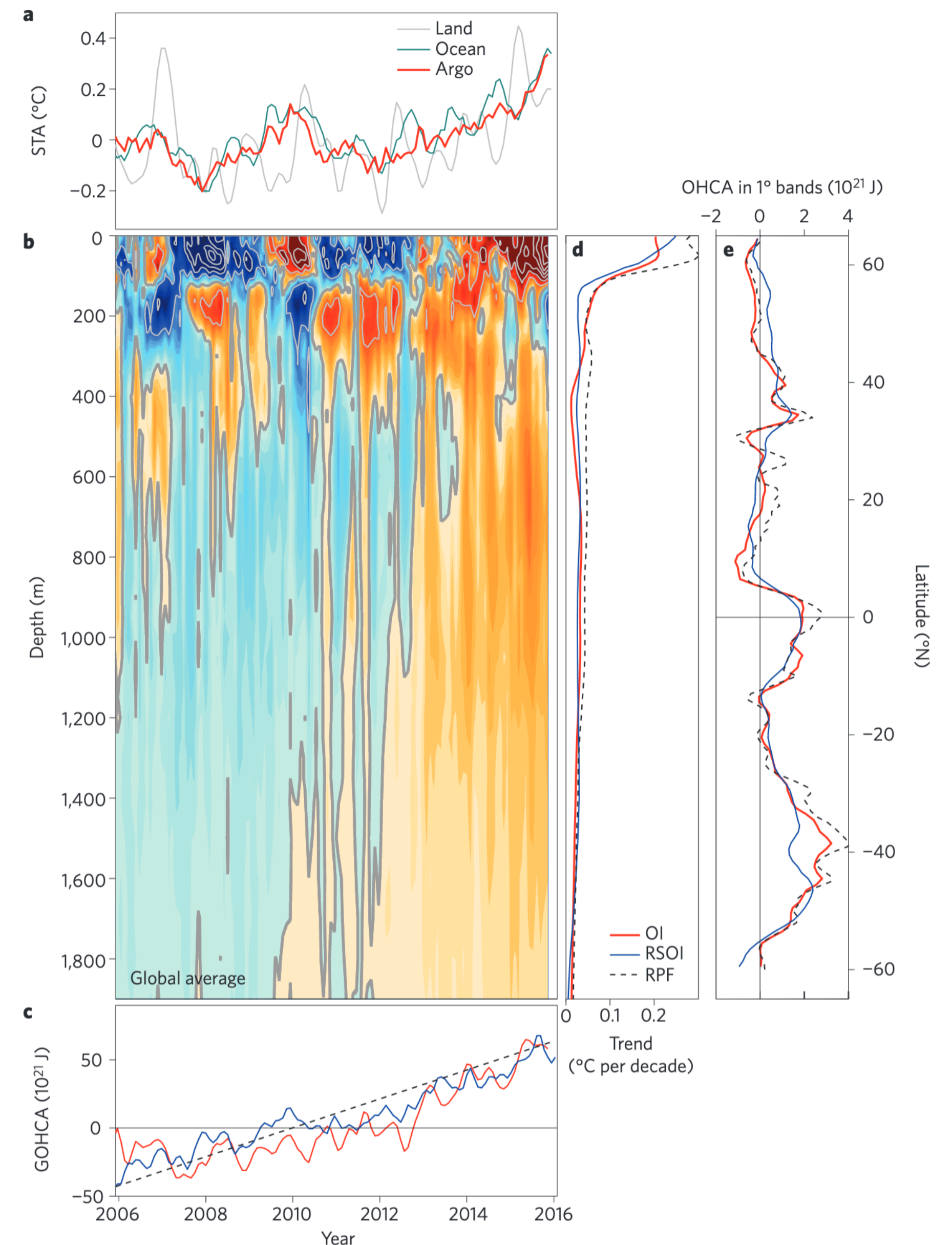
Download two skew-Ts and discuss how the dew point changes with height.

What do you see that catches your attention?



# Last Class: Paper discussion

1. Why is most of the global heat imbalance get stored in the ocean?
2. How do the ARGO observations compare to climate model projections?
3. What does Fig 1a tell us?
4. What about Fig 1b and d?
5. Discuss Fig. 1c
6. Where is the largest rate of oceanic heating occurring? Discuss Fig. 1e
7. What does this all mean for climate change?



## Last Class: Water vapor

Water vapor is roughly an ideal gas. It follows Dalton's law of partial pressures (the total pressure is the sum of the pressure of all the constituent gases).

$$e\alpha_v = R_v T$$

The mixing ratio is the amount of water vapor mass per unit of dry air

$$r_v = \frac{M_v}{M_d}$$

The specific humidity is the amount of water vapor per unit of total air mass.

$$q_v = \frac{M_v}{M_d + M_v} \quad q_v \simeq r_v$$

# Water vapor

Using the ideal gas law we can express the mixing ratio and specific humidity in terms of pressure

$$e = \rho_v R_v T \quad p = \rho R_d T \quad p_d = \rho_d R_d T$$

Which are written as

$$r_v \simeq \varepsilon \frac{e}{p_d} \quad q_v \simeq \varepsilon \frac{e}{p_d + e} \simeq \varepsilon \frac{e}{p}$$

$$\varepsilon = R_d / R_v = 0.622$$



Air only has a limited capacity to hold water vapor. How close you are to this maximum capacity is expressed in terms of the relative humidity.

$$\text{RH} = \frac{e}{e_s}$$

**Enthalpy** version of the first law

$$dh = \delta q + \alpha dp$$

$$h = c_v T + p\alpha \quad \text{The enthalpy}$$

$$h = c_p T \quad \text{For an ideal gas}$$

This is the version of the first law that is used in isobaric coordinate systems, very commonly used in atmospheric sciences



**Gibbs** version of the first law

$$dG = vdp - sdT$$

$$G = h - sT \quad \text{The Gibbs free energy.}$$

Maximum work that can be done by a system at constant temperature and pressure

What advantages might this equation have over other forms of the first law?

We are specifically interested in isobaric, isothermal processes

$$dG = 0$$

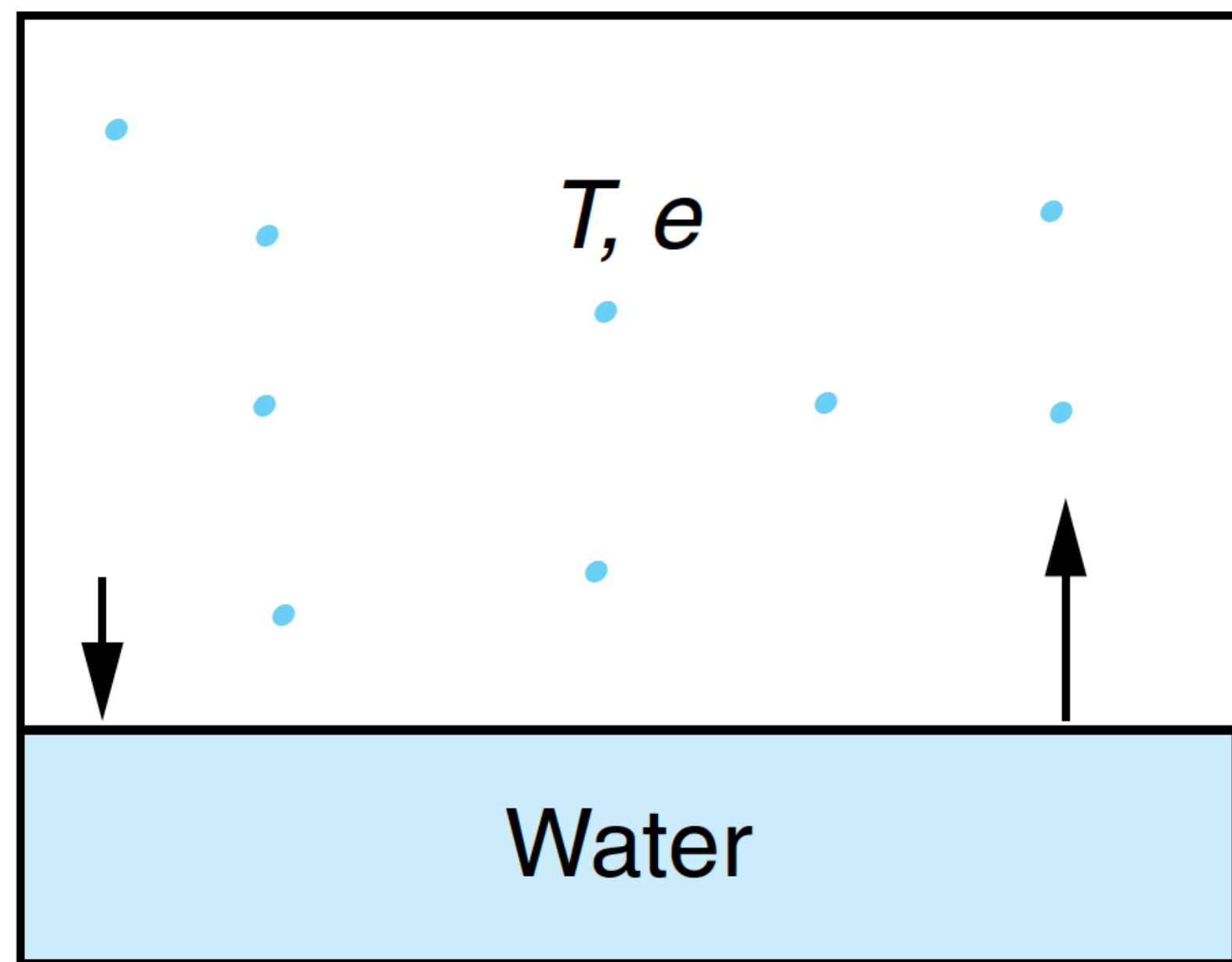
Let's think of a system in which contains liquid water and water vapor, and is whose temperature and pressure remain constant.



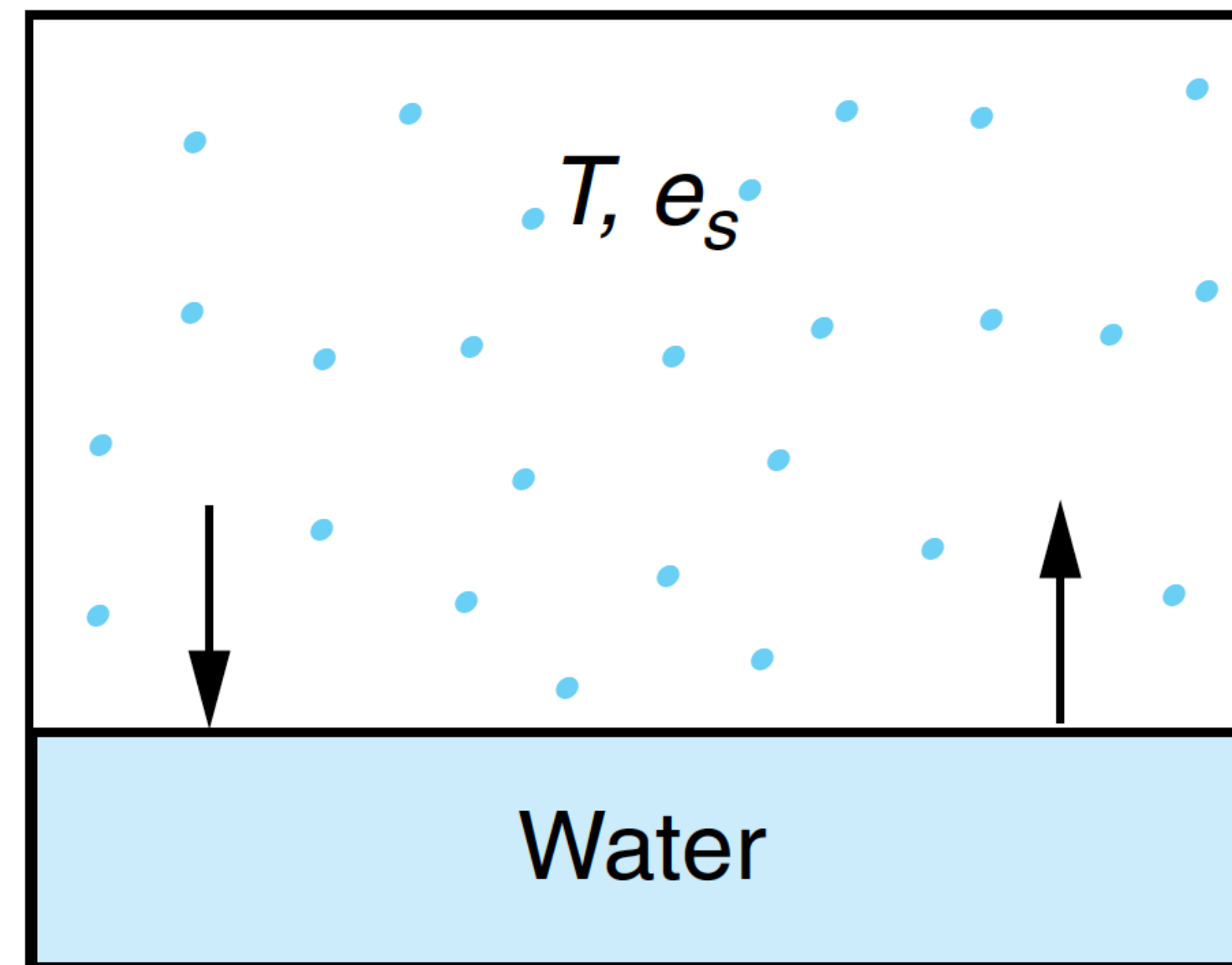
# Gibbs free energy

Imagine you have a tank that is completely closed. The tank has liquid water, dry air and water vapor. Evaporation and condensation happen simultaneously in it, but one rate may exceed the other initially.

Eventually the liquid and the gas reach thermodynamic equilibrium.



(a) Unsaturated



(b) Saturated

*Wallace and Hobbs (2006)*

# Gibbs free energy

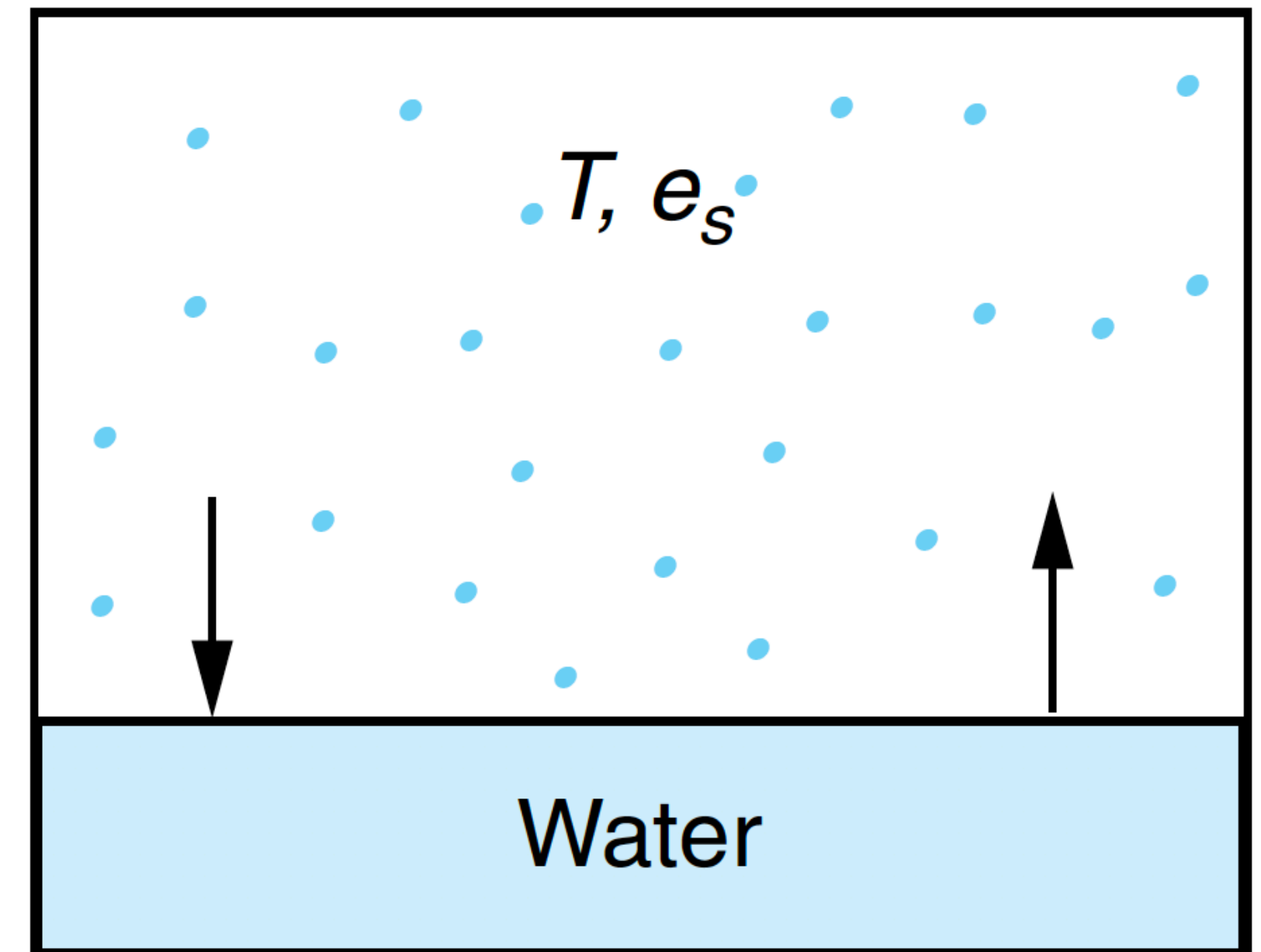
The vapor pressure for equilibrium is known as the saturation vapor pressure  $e_s$

In this state of equilibrium, the tank satisfies

$$\frac{DG}{Dt} = 0$$

Which means that the change in  $G$  for evaporation and condensation must be equal

$$G_v = G_l$$



(b) Saturated

*Wallace and Hobbs (2006)*

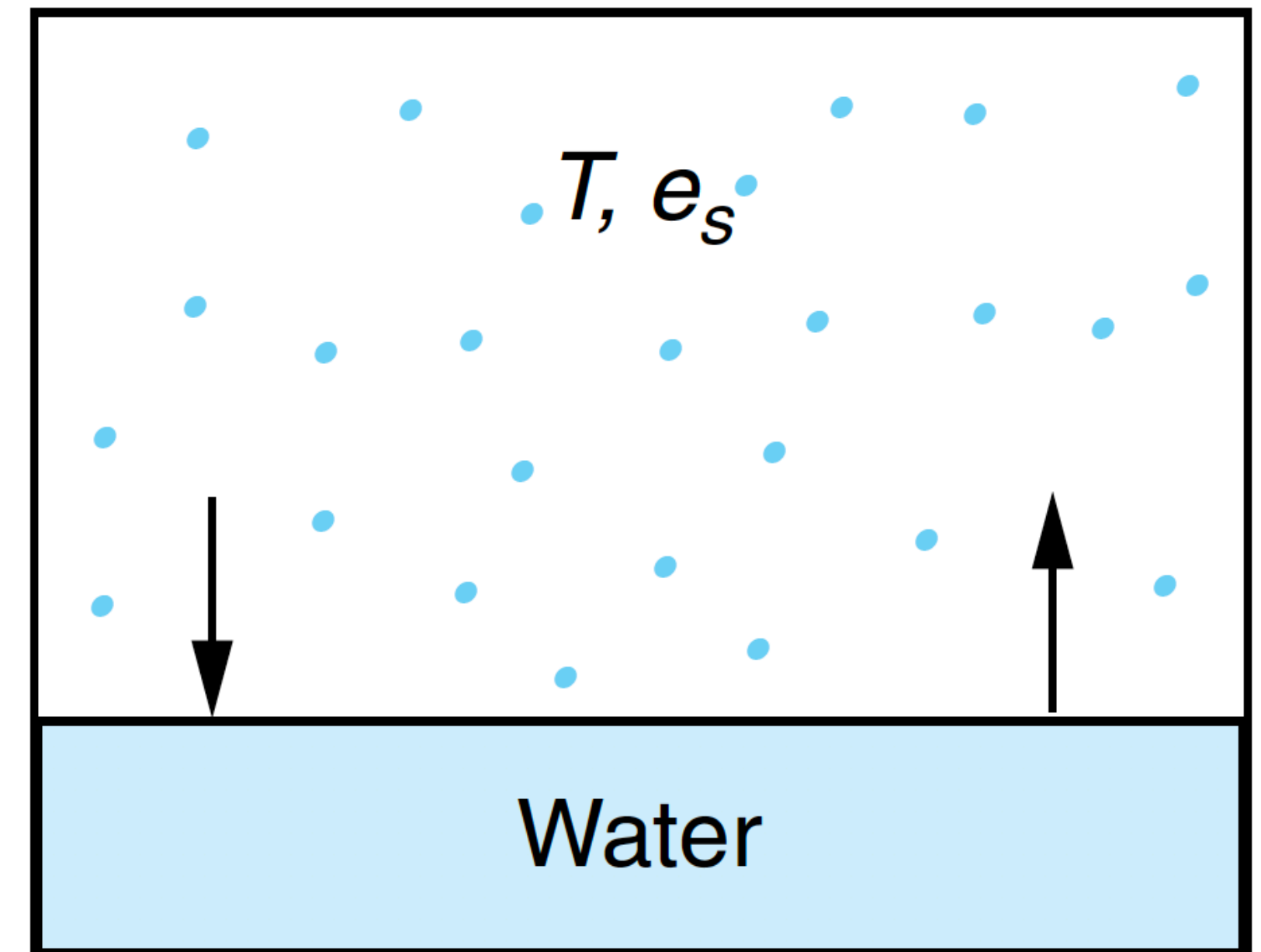


The vapor pressure for equilibrium is known as the saturation vapor pressure  $e_s$

$$\text{RH} = \frac{e}{e_s}$$

If the RH is 100%, you have reached equilibrium. That is, the rate of evaporation to condensation is the same\*

(\*This only applies for flat surfaces of water.  
The story is different for cloud droplets)



(b) Saturated

*Wallace and Hobbs (2006)*

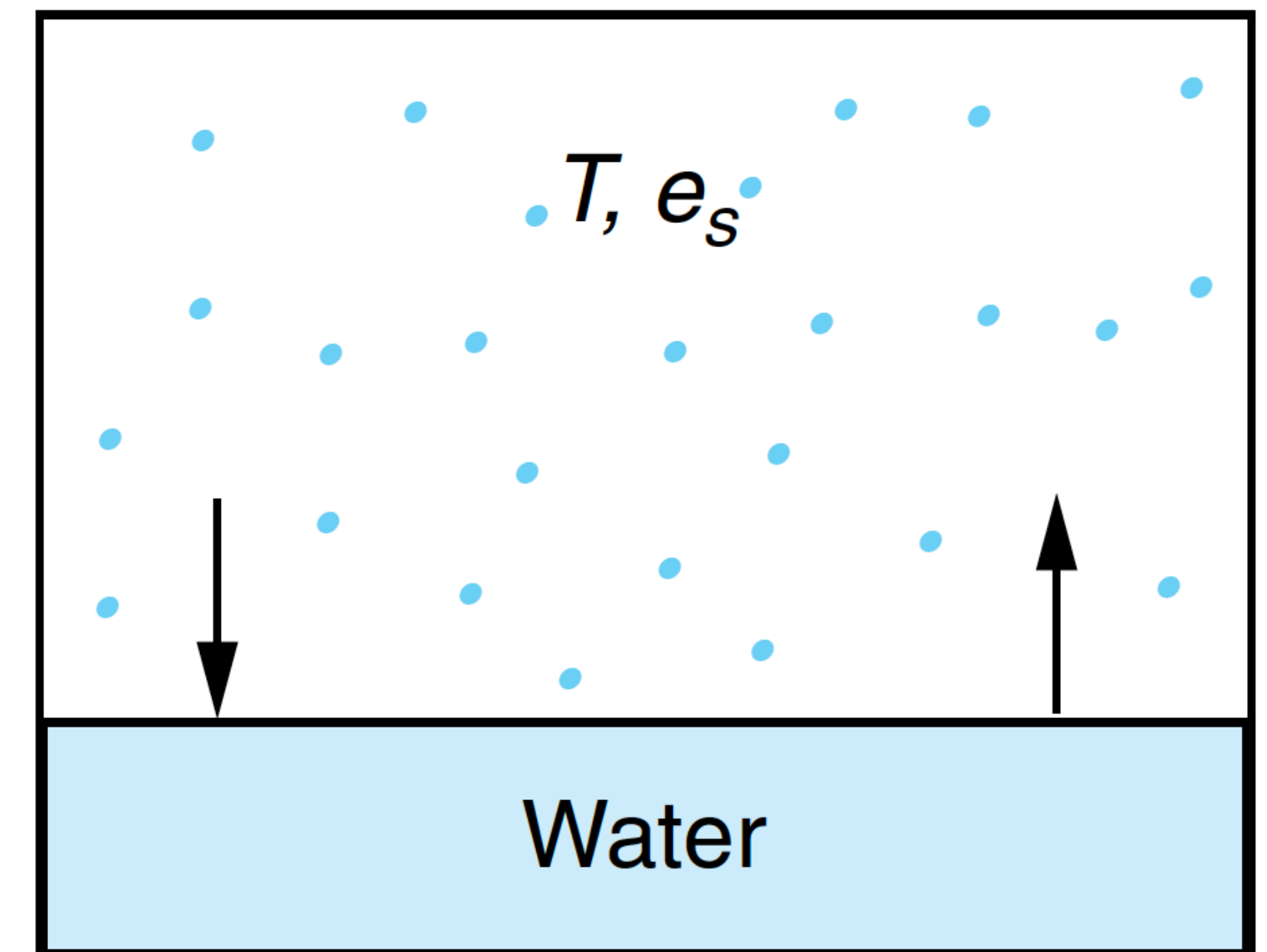
# The Clausius–Clapeyron equation

This means that if the tank is warmed for example,  $G$  in both systems must change equally when equilibrium is reached again.

$$dG_v = dG_l$$

Solving this equality and taking the limit yields the Clapeyron equation

$$\frac{de_s}{dT} = \frac{s_v - s_l}{\alpha_v - \alpha_l}$$



(b) Saturated

*Wallace and Hobbs (2006)*



# Latent energy of vaporization

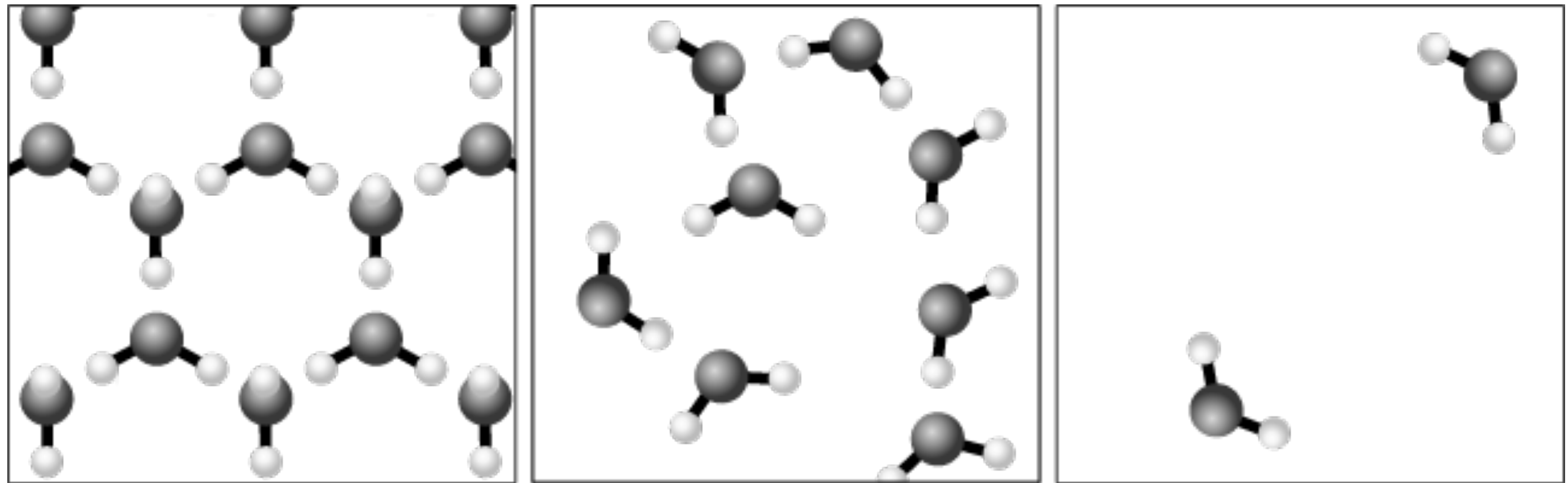
Written in terms of entropy

$$L_v = T(s_v - s_l)$$

Physically-speaking,  $L_v$  describes the increase in enthalpy that the system obtains from the breakage of the molecular bonds in liquid.

If thought as a potential energy (PE), then you can think of the following conservation

$$(PE + h)_l = (PE + h)_v$$



# The Clausius–Clapeyron equation: solution

Plugging the definition of  $L_v$  into the Clapeyron equation yields the Clausius-Clapeyron Equation.

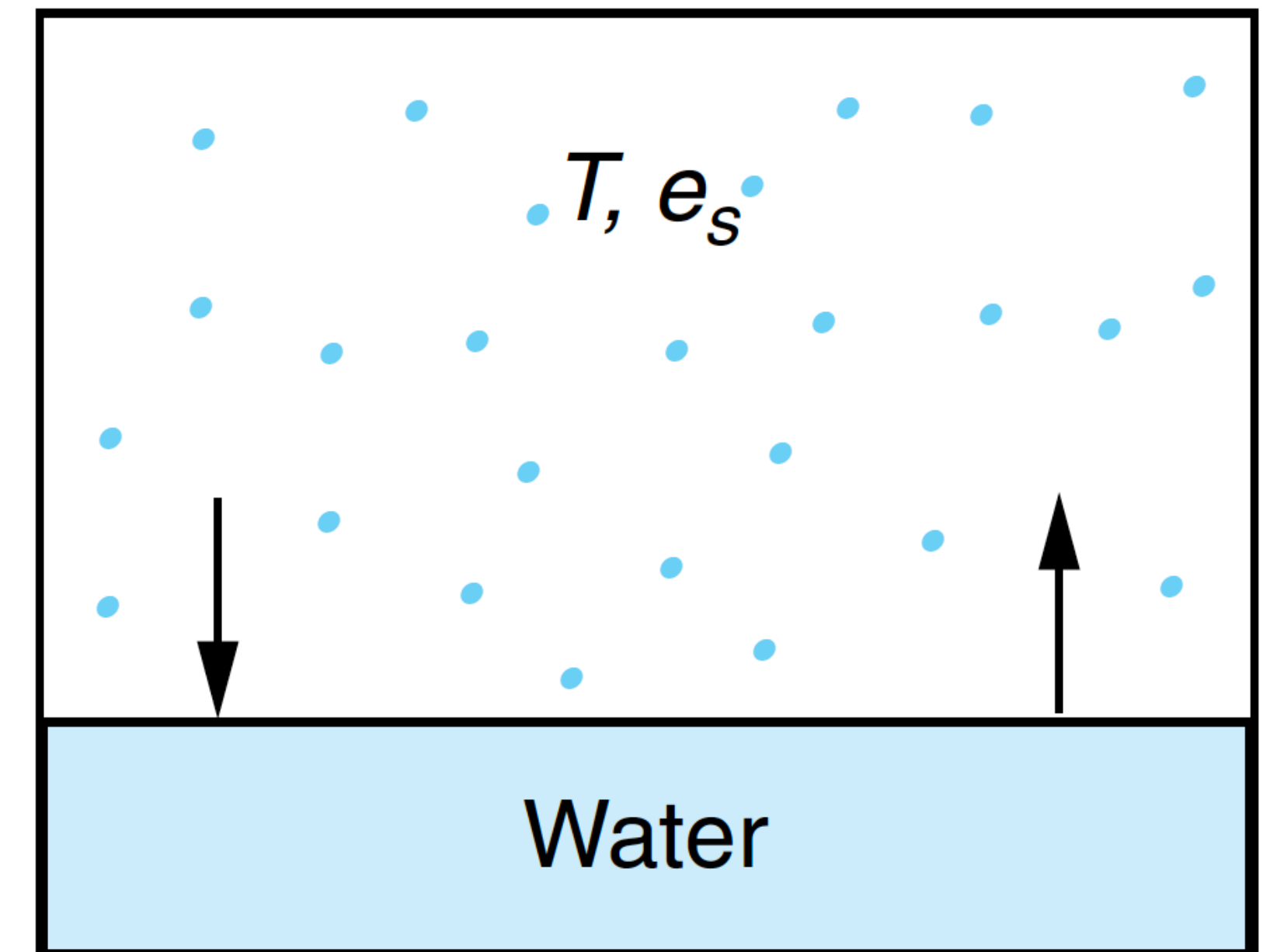
$$\frac{de_s}{dT} = \frac{L_v}{T(\alpha_v - \alpha_l)}$$

Assuming that

$$\alpha_v \gg \alpha_l$$

Yields a solution of the form

$$e_s \simeq e_{s0} \exp \left( \frac{L_v}{R_v} \left[ \frac{1}{T_0} - \frac{1}{T} \right] \right)$$



(b) Saturated

*Wallace and Hobbs (2006)*

# The Clausius–Clapeyron equation: solution

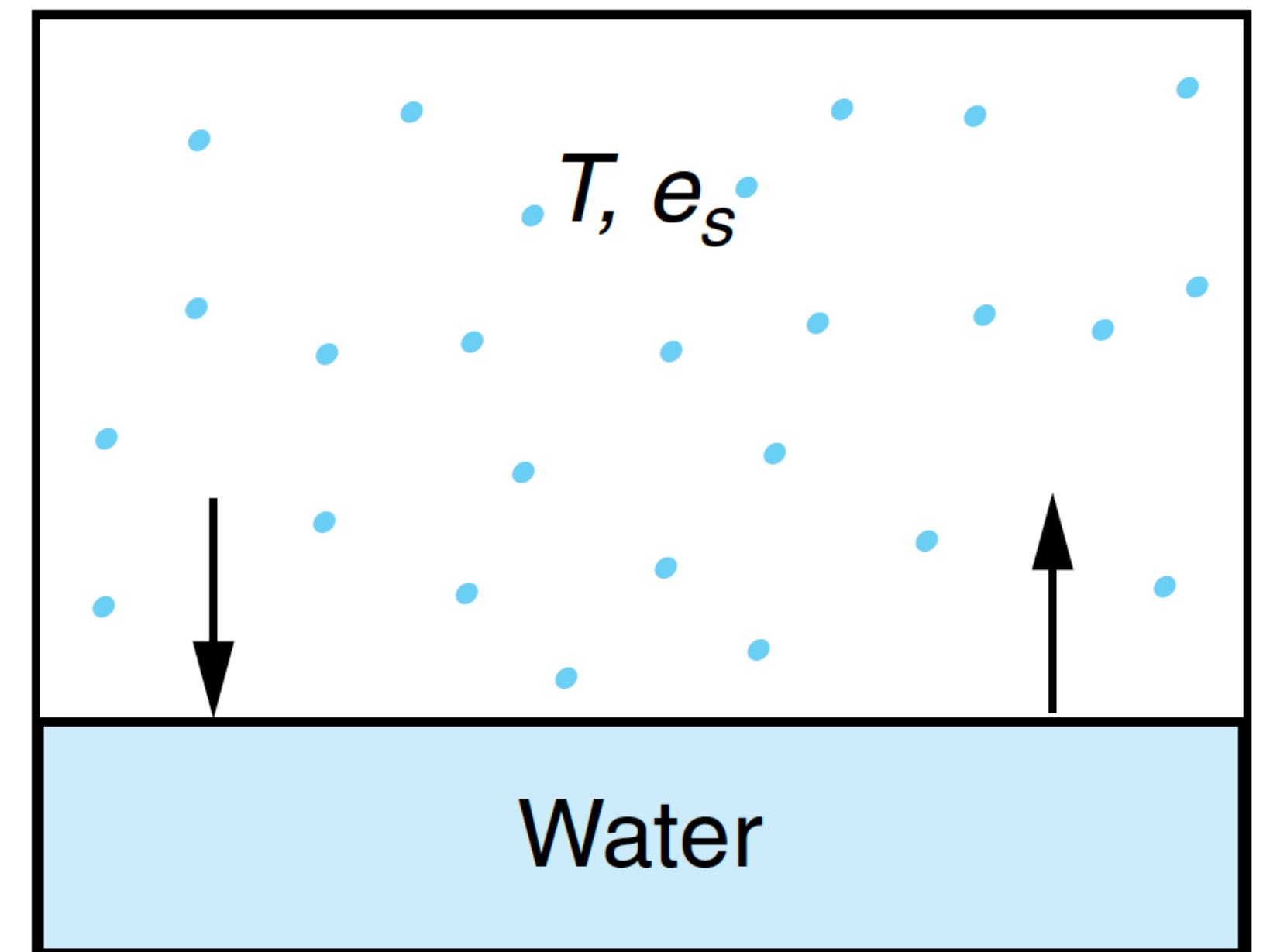
A simplified version of the solution takes the following form

$$e_s = A \exp\left(-\frac{B}{T}\right)$$

$$A = 2.53 \times 10^9 \text{ hPa}$$

$$B = L_v/R_v = 5.42 \times 10^3 \text{ K}$$

This is a nice version that you can use to obtain other variables that are related to the Clausius–Clapeyron equation.



(b) Saturated

*Wallace and Hobbs (2006)*



Is the temperature in which an unsaturated air parcel will reach saturation if we adiabatically cool it and keep the mixing ratio and the pressure constant.

$$e(T_d) = e_s$$

Can use Clausius-Clapeyron to obtain it

$$T_d(p, q_v) = \frac{B}{\ln \left( \frac{A\varepsilon}{pq_v} \right)}$$

$$A = 2.53 \times 10^9 hPa$$

$$B = L_v/R_v = 5.42 \times 10^3 K$$

