AOS 630: Introduction to Atmospheric and Oceanic Physics
Lecture 3 Fall 2021
Moist air and hydrostatic balance

Ángel F. Adames-Corraliza
angel.adamescorraliza@wisc.edu

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Announcements

Homework 1 is now online. It is due two weeks from today.

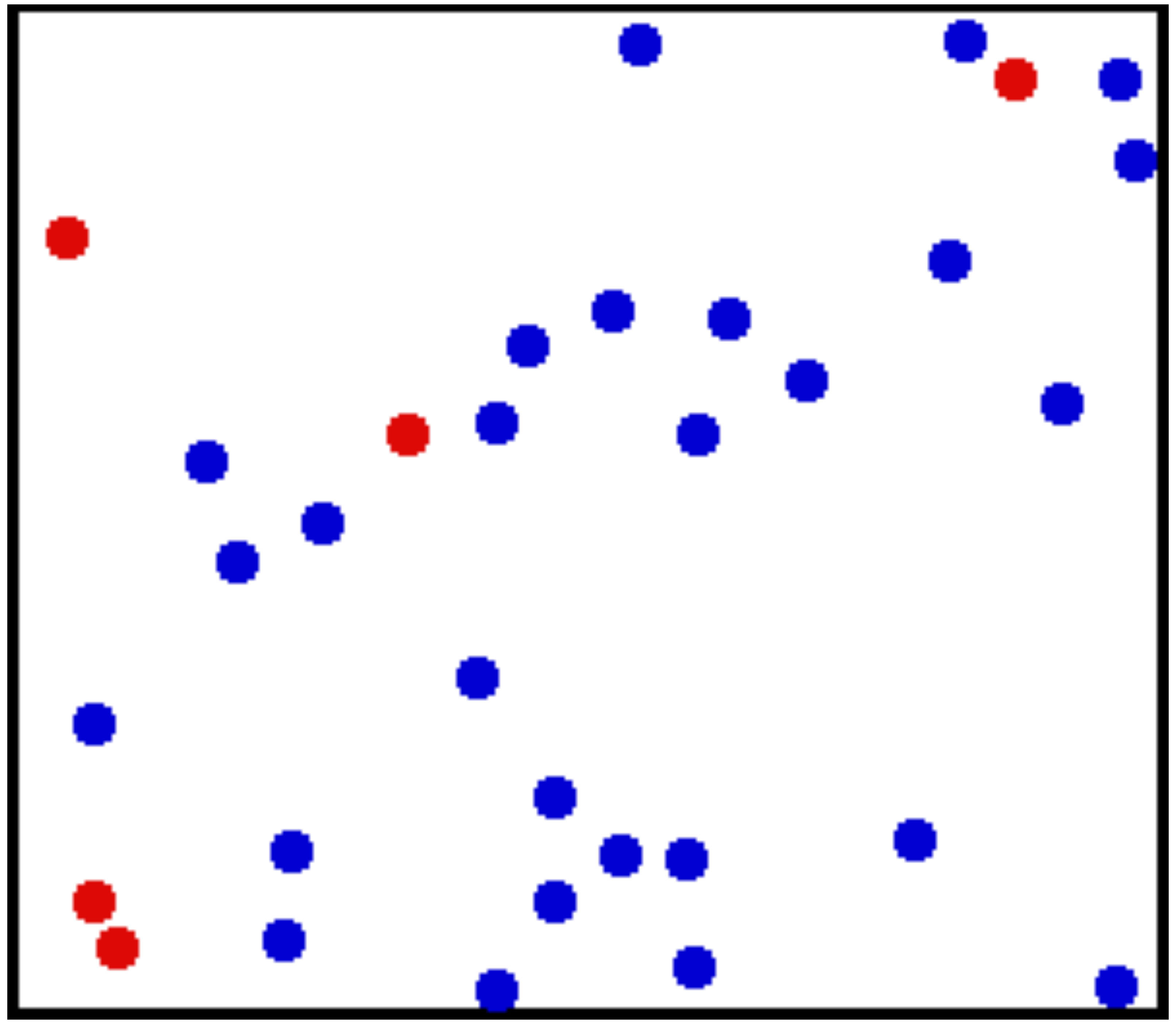
Last Class: the equation of state

For dry air

$$p_d = \rho_d R_d T$$

$$R_d = 287 \mathbf{J\ kg}^{-1}\mathbf{K}^{-1}$$

Dry gas constant



Partial pressure for water vapor

$$e = \rho_v R_v T$$

$$R_v = 461 \mathbf{J\ kg}^{-1}\mathbf{K}^{-1}$$

Water vapor constant

Partial pressure for moist air

$$p = p_d + e$$

$$p = (\rho_d R_d + \rho_v R_v) T$$

Equation of state for seawater

$$\rho = \rho_0 \left[1 - \beta_T \left(T - T_0 \right) + \beta_s \left(S - S_0 \right) + \beta_p \left(p - p_0 \right) \right]$$

Symbol	Description	Value
ρ_0	reference density	1.027×10^3 kg m^{-3}
α_0	reference specific volume	9.738×10^{-4} m^3 kg^{-1}
T_0	reference temperature	283 K
S_0	reference salinity	35 ppt = 35 g kg^{-1}
c_{s0}	reference sound speed	1490 m s^{-1}
β_T	thermal expansion coefficient	1.67×10^{-4} K^{-1}
β_S	haline contraction coefficient	0.78×10^{-3} ppt^{-1}
β_p	compressibility coefficient ($= \alpha_0 / c_{s0}^2$)	4.39×10^{-10} m s^2 kg^{-1}
c_{p0}	specific heat capacity at const. pressure	3986 J kg^{-1} K^{-1}

This is just an approximation. The equation that is used in ocean modeling is much more complicated!

Today

Examine how water vapor modifies the equation of state

Introduce the hydrostatic equation

Supplementary reading

Petty
Sections 3.4 and Chapter 4

Wallace and Hobbs
Section 3.1.1 and 3.2

Water vapor

Water vapor is roughly an ideal gas. It follows Dalton's law of partial pressures (the total pressure is the sum of the pressure of all the constituent gases).

$$e\alpha_v = R_v T$$

The mixing ratio is the amount of water vapor mass per unit of dry air

$$r_v = \frac{m_v}{m_d}$$

The specific humidity is the amount of water vapor per unit of total air mass.

$$q_v = \frac{m_v}{m_d + m_v} \qquad q_v \simeq r_v$$

Water vapor

Using the ideal gas law we can express the mixing ratio and specific humidity in terms of pressure

$$e = \rho_v R_v T \qquad\qquad p = \rho R_d T$$

Which are written as

$$r_v \simeq \varepsilon \frac{e}{p} \qquad\qquad q_v \simeq \varepsilon \frac{e}{p + e}$$

$$\varepsilon = R_d / R_v = 0.622$$

Modifications of the equations due to moisture

Water vapor is lighter that the dry air molecules O_2 and N_2

This means that a volume of humid air that is at the same temperature as a volume of dry air is actually less dense

Constituent[a]	Molecular weight	Fractional concentration by volume
Nitrogen (N_2)	28.013	78.08%
Oxygen (O_2)	32.000	20.95%
Argon (Ar)	39.95	0.93%
Water vapor (H_2O)	18.02	0–5%
Carbon dioxide (CO_2)	44.01	380 ppm
Neon (Ne)	20.18	18 ppm
Helium (He)	4.00	5 ppm
Methane (CH_4)	16.04	1.75 ppm
Krypton (Kr)	83.80	1 ppm
Hydrogen (H_2)	2.02	0.5 ppm
Nitrous oxide (N_2O)	56.03	0.3 ppm
Ozone (O_3)	48.00	0–0.1 ppm

Virtual Temperature

To take into account this change in density we define the virtual temperature

$$p\alpha = R_d T_v$$

$$T_v \simeq T\left(1 + 0.61 q_v\right)$$

The virtual temperature is the temperature dry air would have if it had the same density as the moist air at the same pressure.

Exercise

Calculate the virtual temperature for the locations below.

Based on your answer, do you think the virtual temperature correction may be important somewhere and why?

$$T_v \simeq T\left(1 + 0.61 q_v\right)$$

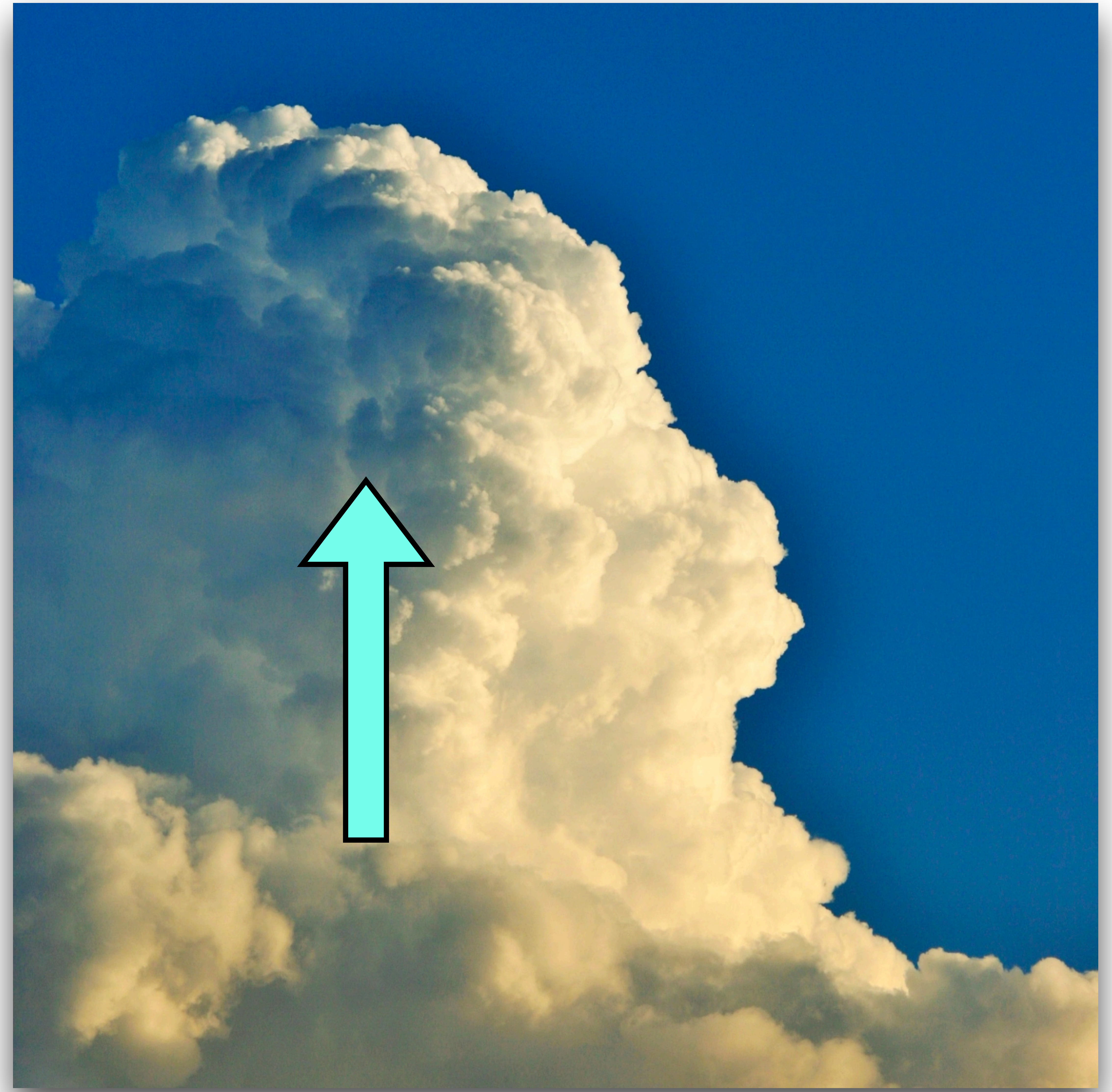
Location	Temperature (°C)	Mixing ratio (g/kg)
Utqiaġvik (Barrow), AK	0	5
Gaylord, MI	15	10
Singapore	30	30

Newton's second law dictates that acceleration must result from a net sum of forces.

Apply this to vertical motion

$$\frac{Dw}{Dt} = \frac{1}{m}\Sigma_i F_z$$

$$\frac{D}{Dt} = \frac{d}{dt}$$



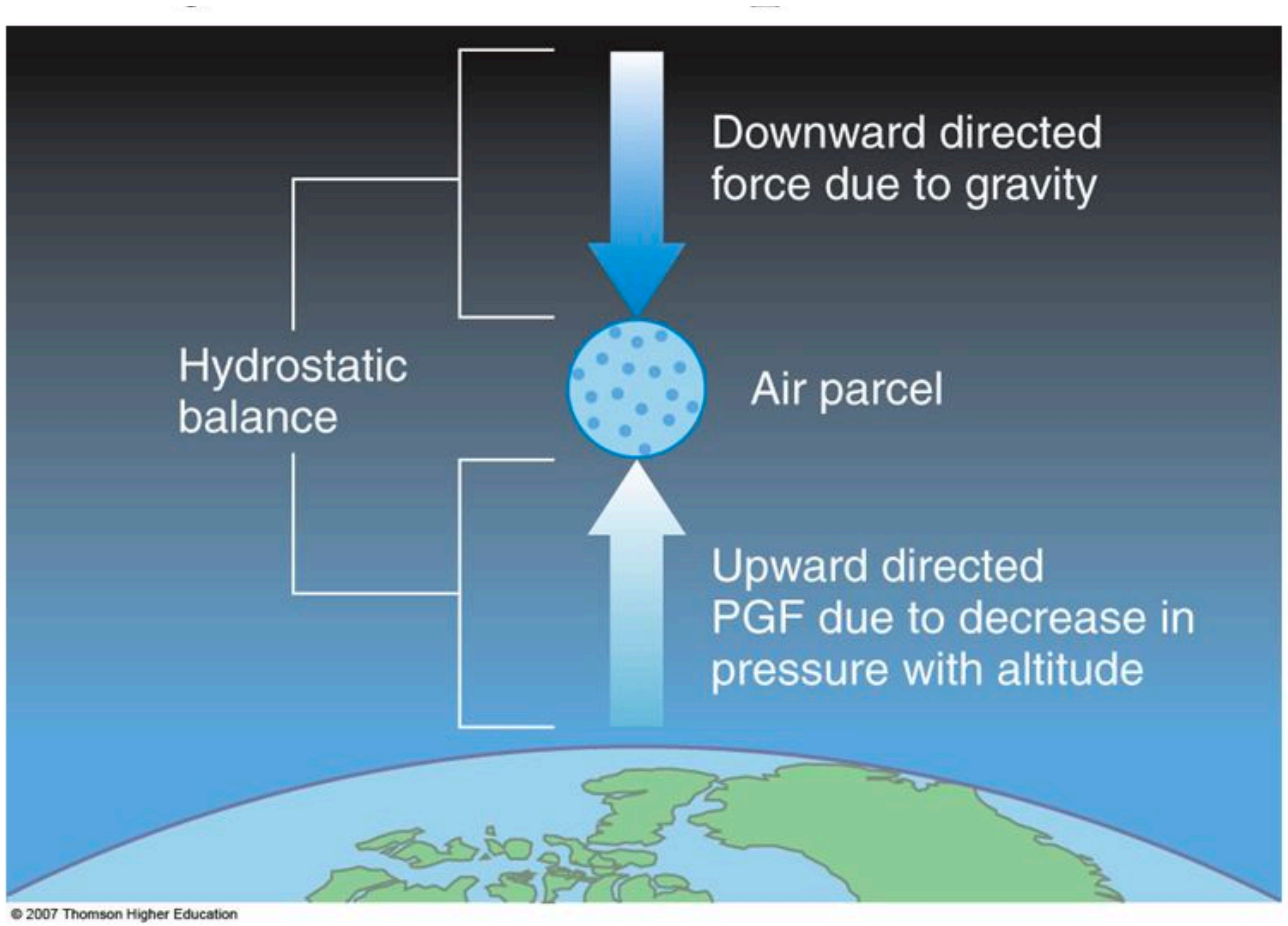
Ignoring the effects of planetary rotation and friction, the two main forces that cause vertical acceleration are gravity and the pressure gradient force.

$$\frac{Dw}{Dt} = -\alpha\frac{\partial p}{\partial z} - g$$

Acceleration

Pressure gradient force

Gravity



Hydrostatic Balance

For quiescent atmospheric conditions, the atmosphere is maintained in place by a balance between the **downward gravitational force** and the **upward pressure gradient force.**

$$\rho g \simeq -\frac{\partial p}{\partial z}$$

Using ideal gas law we can obtain profiles for how pressure changes with height

$$p\alpha = R_d T_v$$

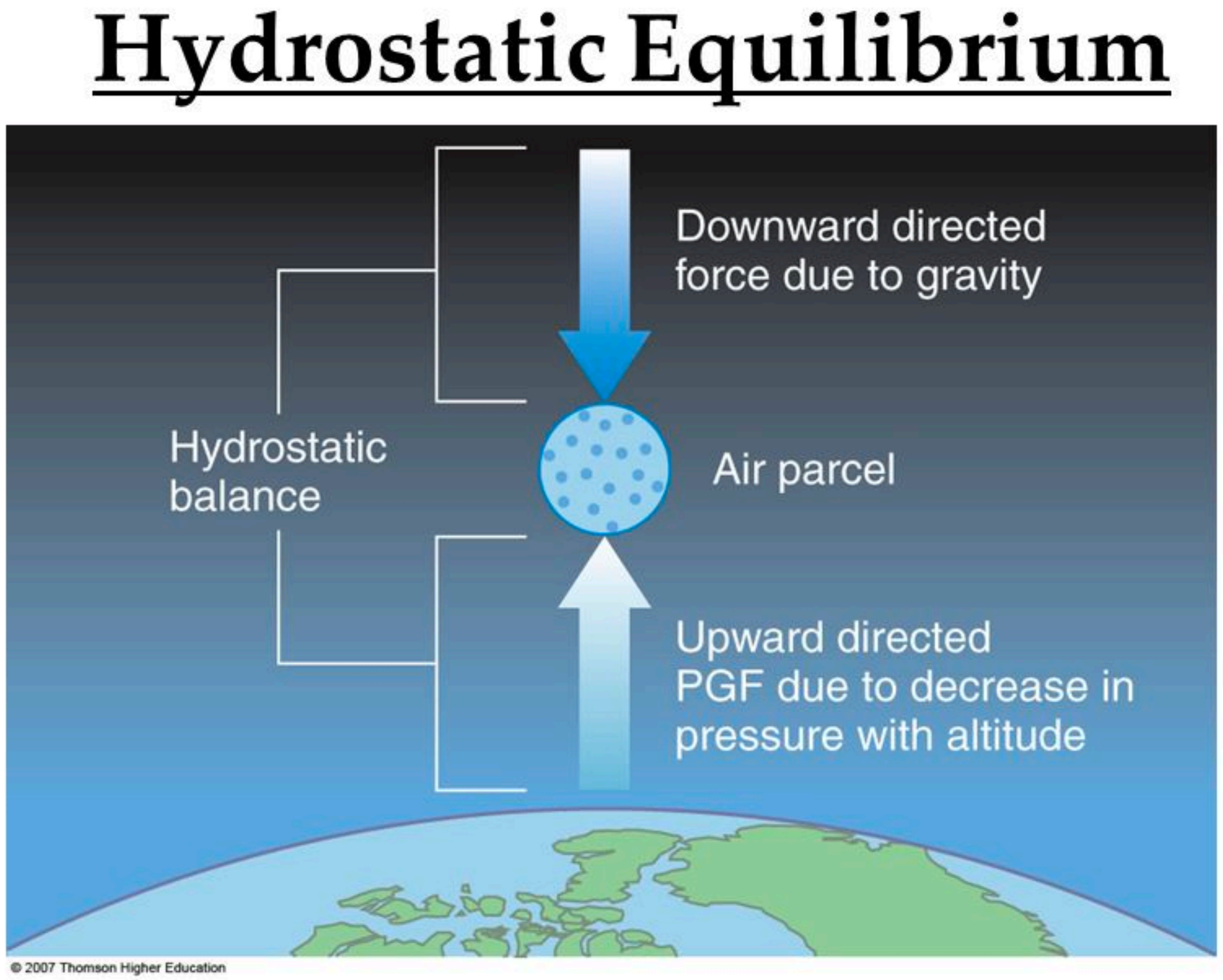


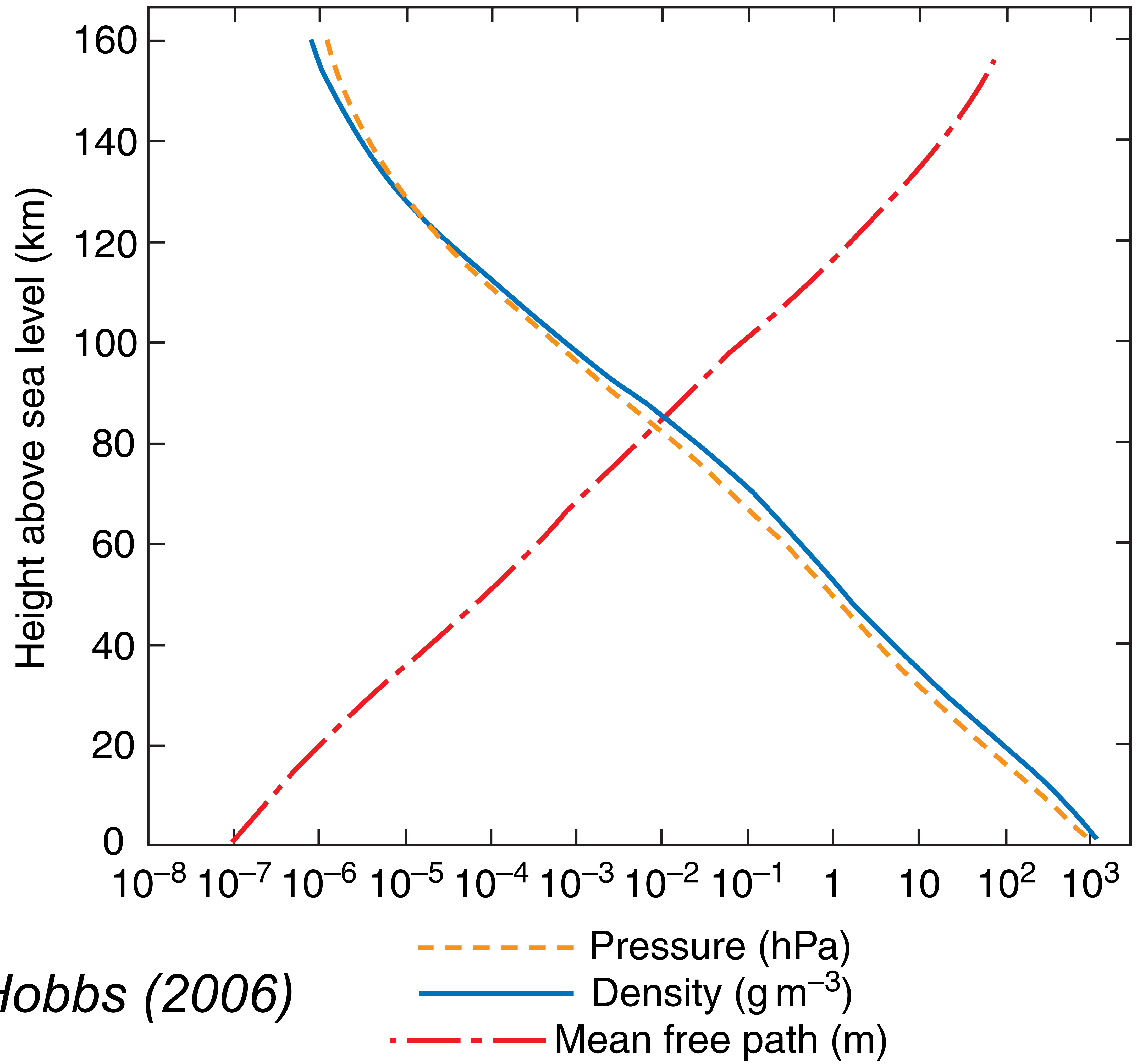
Fig. 6-13, p. 171

Example

For an isothermal atmosphere, can easily solve the differential equation to obtain

$$p \simeq p_0 \exp\left(-\frac{gz}{R_d \overline{T}_v}\right)$$

$\overline{T}_v$ Mean virtual temperature of troposphere



Wallace and Hobbs (2006)

Hypsometric equation

For smaller layers of the atmosphere, we can also solve the hydrostatic equation to obtain the “thickness” equation

$$Z_2 - Z_1 = \frac{R_d}{g_0} \int_{p_2}^{p_1} T_v \frac{dp}{p}$$

Which we can simplify by replacing the virtual temperature with its layer mean value to obtain

$$Z_2 - Z_1 = \overline{H} \ln\left(\frac{p_1}{p_2}\right) = \frac{R_d \overline{T}_v}{g_0} \ln\left(\frac{p_1}{p_2}\right)$$

Which is known as the hypsometric equation.

How about the ocean?

$$\rho = \rho_0 \left[1 - \beta_T \left(T - T_0 \right) + \beta_s \left(S - S_0 \right) + \beta_p \left(p - p_0 \right) \right]$$

To a good approximation we can treat seawater as incompresible

$$\rho \simeq \rho_0$$

We can solve the hydrostatic equation to obtain the following:

$$p = p_0 + \rho_0 g z$$